Abstract

For deep-space missions that remain in the vicinity of a target body, solar eclipses might become a source of major concern due to thermal and power constraints. This paper presents a method for minimizing the eclipse duration of periodic trajectories in three-body systems using synodic resonant periodic orbits. The proposed methodology successfully captures the global eclipse structure of synodic resonant periodic orbits in terms of driving parameters such as elongation and phase angles. Two-dimensional Eclipse maps are introduced to identify optimal orbit insertion conditions that avoid or minimize eclipse intervals. The validity and applicability of the proposed method is tested in the full-ephemeris model (DE430) of the Earth-Moon and Mars-Phobos systems, respectively. As a result of these investigations, we propose new science eclipse-minimum trajectories that are under consideration for the upcoming JAXA missions EQUULEUS and MMX.

Keywords: Circular restricted three-body problem (CR3BP), Eclipse,
1. Introduction

Solar eclipses have a significant impact on the design of spacecraft that rely on incident Sun light for power generation and thermal control. As a result, the eclipse problem is a major concern for the space community, and especially for deep-space explorers that are constantly looking for innovative breakthroughs to cope with tight engineering constraints.

JAXA’s Venus explorer Akatsuki \[1\] failed to inject into its originally planned orbit due to a malfunction in the propulsion system. While an alternative orbit around Venus was found, it contained unacceptable long eclipses that endangered the safety of the spacecraft. It was not until the selection of proper initial conditions and tight phase control actions that long-term eclipses could finally be avoided and operations resumed \[2\]. Gaia \[3\], the European mission to build a 3D map of the MilkyWay, had also encountered the eclipse problem during the preliminary trajectory design of Lissajous orbits around the Sun-Earth Lagrangian point \(L_2\). In this case, adding additional maneuvers to change the shape and phase of the Lissajous orbits was necessary in order to avoid eclipses and perform scientific observations \[4\]. Other several proposed missions have encountered similar eclipse problems, such as JUICE \[5\], LUMIO \[6\], and JWST \[7\]. A similar situation occurred to the authors of this paper while designing science orbits for the CubeSat mission EQUULEUS \[8\].

In the literature, several researchers have pursued solutions to mitigate these kind of eclipse problems. Eclipse analysis on trajectories in three-body dynamics has been performed by Pergola et al. \[9\] and Tang et al. \[10\], who first catalogued the population of CR3BP periodic orbits affected by eclipses in the Earth-Moon system. Meanwhile, Canalias et al. published an eclipse-mitigation strategy based on impulsive maneuvers along lissajous orbits in the Sun-Earth system \[11\]. More recently, a new line of work on eclipse mitigation for periodic orbits in the Earth-Moon system has been developed, following NASA’s plans
to return humans to the surface of the Moon \cite{12}. These recent developments are based on periodic orbits whose period is resonant with the synodic month of the Moon. Such orbits are known in the literature as Synodic Resonant Periodic Orbits (SR-PO) and were investigated by Williams et al. \cite{13}, Zimovan et al. \cite{14,15}, McCarthy et al. \cite{16} and Boudad et al. \cite{17}. The above contributions are typically centered on SR-halo orbits (SR-HO) and SR-Distant Retrograde Orbits (SR-DRO) in the Earth-Moon system because of their relevance for Orion and the upcoming lunar orbital platform gateway \cite{18}. Because of their favorable eclipse conditions and stability properties, near-rectilinear SR-HO are currently being considered for future lunar exploration activities \cite{19}. Nonetheless, the systematic approach that explains minimum-eclipse trajectories has not been investigated, even though similar eclipse conditions can be found under different three-body systems.

This paper proposes a general approach to design minimum-eclipse trajectories in three-body systems through analytical developments and geometrical insight. Building on the author’s previous results \cite{20}, this investigation derives new analytical formulas for identifying driving parameters for eclipse minimization and design new scientific orbits for the upcoming JAXA missions EQUULEUS and MMX \cite{21,22}. To this end, this paper introduces two-dimensional maps that help mission designers identify minimum-eclipse time orbit insertion conditions for their missions and verify our assumptions under the real-ephemeris dynamics (DE 430) of the Earth-Moon and Mars-Phobos systems, respectively.

2. Background

This section presents the equations of motion that model the Circular Restricted Three-Body Problem (CR3BP) and a coordinate transformation for eclipse visualization that is important for the later discussion in Sec. 3. Following that, synodic period and SR-PO, that are important for eclipse minimization are introduced.
2.1. Dynamical Model

The CR3BP framework is used to describe the motion of a spacecraft in three-body systems. Two following assumptions are made for derivation of equations of motion, as shown in Fig. [1]

1. The mass of the third body satisfies $m_1 > m_2 >> m_3$,
2. $P_1$ and $P_2$ move in circular orbits about their common center of mass.

Under these assumptions, the equations of the CR3BP in normalized units become

\[
\begin{cases}
\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}, \\
\ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}, \\
\ddot{z} = \frac{\partial U^*}{\partial z},
\end{cases}
\]

where $x, y, z, \dot{x}, \dot{y}, \dot{z}$ denote the position and velocity components of a spacecraft in the rotating frame of Fig. [1] $U^*$ corresponds to the effective potential

\[
U^* = \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}(x^2 + y^2),
\]
where \( r_i \) is the distance between \( P_i \) and \( P_3 \), and \( \mu \) is the mass ratio parameter, defined as
\[
\mu = \frac{m_2}{m_1 + m_2}.
\]
Multiplying Eq. (1) by \( \dot{x}, \dot{y}, \dot{z} \), respectively, and summing over all of the terms yields the Jacobi constant \( C \), which is the only integral of motion of the CR3BP:
\[
2U^* - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = C.
\]

The position of the Sun in the synodic frame of the two primaries needs to be computed for eclipse purposes. In this research, it will be initially assumed that the motion of the Sun is co-planar with the motion of the two primaries and governed by the equations of the two-body problem between the Sun and a virtual spherical object of mass \((m_1 + m_2)\), located at the origin of Fig. 1. As seen from the synodic frame of \( P_1 \) and \( P_2 \), i.e. CR3BP frame, the Sun moves around the barycenter in a retrograde circular orbit described by
\[
\mathbf{r}_\odot = a_\odot \begin{bmatrix} \cos \zeta(t) \\ \sin \zeta(t) \\ 0 \end{bmatrix}, \quad \zeta(t) = \zeta_0 - \Omega t,
\]
where \( \zeta \) is related to the elongation angle \( \Theta \), which is the angle between \( P_1 \)-Sun and \( P_1 \)-\( P_2 \) as shown in Fig. 1, via
\[
\sqrt{r_\odot^2 + (1 - \mu)^2 - 2r_\odot(1 - \mu) \cos \zeta} = \sqrt{(r_{P1}^\odot)^2 + 1 - 2r_{P1}^\odot \cos \Theta},
\]
where \( a_\odot \) is the semi-major axis of the Sun orbit with respect to barycenter in normalized units, \( \Omega = 1 - \omega_\odot \), and \( r_{P1}^\odot = \| \mathbf{r}_{P1}^\odot \| \) is the distance between the Sun and the larger of the two primaries. Following our assumptions, \( \omega_\odot = \sqrt{(GM_\odot + 1)/a_\odot^3} \), where \( GM_\odot \) is the gravitational parameter of the Sun in normalized units. The values of \( \mu \), \( GM_\odot \), \( a_\odot \), \( \omega_\odot \), and \( \Omega \) used in numerical simulations are tabulated in Table 1. Finally, observe that the gravitational influence of the Sun on the dynamical evolution of mass particles subject to the equations of motion (1) is neglected until Sec. 5, where EQUULEUS and MMX trajectories are optimized.
in the real-ephemeris model of the Sun-Earth-Moon and Sun-Mars-Phobos systems, respectively.

<table>
<thead>
<tr>
<th>System</th>
<th>( \mu )</th>
<th>LU (km)</th>
<th>TU (s)</th>
<th>( GM_\odot ) (-)</th>
<th>( a_\odot ) (-)</th>
<th>( \omega_\odot ) (-)</th>
<th>( \Omega ) (-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth-Moon</td>
<td>0.01215</td>
<td>384400</td>
<td>375190</td>
<td>328900</td>
<td>389</td>
<td>0.07470</td>
<td>0.9253</td>
</tr>
<tr>
<td>Mars-Phobos</td>
<td>( 1.655 \times 10^{-8} )</td>
<td>9376</td>
<td>4387</td>
<td>3098703</td>
<td>24311</td>
<td>4.644 \times 10^{-4}</td>
<td>0.9995</td>
</tr>
</tbody>
</table>

2.2. Coordinate Transformation

In the following section, the position vector of the spacecraft \( \mathbf{r} = [x, y, z]^T \) is projected into the \( P_1 \)-Sun and \( P_2 \)-Sun rotating reference frames in order to simplify eclipse analysis as shown in the Fig. 1. According to this convention, the positive \( x \)-axes of the rotating frames are constantly aligned with the body-Sun relative position vectors, implying that the shadow cone emanating from the occulting body is fixed along the negative \( x \)-axis. Denoting \( \mathbf{r}^A_B \) as the relative position vector from point \( A \) to \( B \), one finds that

\[
\mathbf{r}^{P_1}_\odot = \mathbf{r}_\odot - \mathbf{r}_{P_1}, \quad \mathbf{r}^{P_2}_\odot = \mathbf{r}_\odot - \mathbf{r}_{P_2},
\]

with \( \mathbf{r}_{P_1} = [-\mu, 0, 0]^T \) and \( \mathbf{r}_{P_2} = [1-\mu, 0, 0]^T \), respectively. Assuming \( \mathbf{Z}^{P_1}_\odot // \mathbf{Z}^{P_2}_\odot // \hat{\mathbf{z}} \), it follows that the body-Sun rotating references frames centered on either of the two primaries are defined by

\[
\mathbf{X}^{P_i}_\odot = \frac{\mathbf{r}^{P_i}_\odot}{|\mathbf{r}^{P_i}_\odot|}, \quad \mathbf{Y}^{P_i}_\odot = \mathbf{Z}^{P_i}_\odot \times \mathbf{X}^{P_i}_\odot, \quad \mathbf{Z}^{P_i}_\odot = \mathbf{\hat{z}}, \quad i = 1, 2.
\]

To find the projection of the spacecraft position vector in these coordinate systems, let us first denote \( \mathbf{r}^{P_i}_S = \mathbf{r} - \mathbf{R}_{P_i} \) as the relative position vectors of the spacecraft when seen from either \( P_1 \) or \( P_2 \). Then, the component of \( \mathbf{r}^{P_i}_S \) in the rotating reference frame of choice can be found through the dot products of \( \mathbf{r}^{P_i}_S \) and the unit vectors of Eq. (8).
2.3. Eclipse Model

For the following eclipse analysis, the conical eclipse model is utilized under the assumption that all of the bodies in our simulations are spherical [23]. Figure 2 depicts the conical eclipse model and vector notations introduced throughout this paper. Two region are introduced depending on the amount of Sun light available to the spacecraft. The spacecraft is in penumbra if it is partially illuminated by the Sun. In contrast, the spacecraft is in the umbra of the occulting body whenever it is completely shadowed by it. When the spacecraft is behind the occulting body,

$$r_{Q1}^S \cdot \hat{X}_{P_i}^S \leq 0$$  

and two equations can be formulated to detect eclipses:

$$\theta_1 = \cos^{-1} \left( \frac{r_{Q1}^S \cdot \hat{X}_{P_i}^S}{r_{Q1}^S} \right) \leq \theta_{1\text{Max}} = \sin^{-1} \left( \frac{R_\odot + R_{P_i}}{r_{P_i}^S} \right),$$  

$$\theta_2 = \cos^{-1} \left( \frac{r_{Q2}^S \cdot \hat{X}_{P_i}^S}{r_{Q2}^S} \right) \leq \theta_{2\text{Max}} = \sin^{-1} \left( \frac{R_\odot - R_{P_i}}{r_{P_i}^S} \right).$$  

In Eq. (10) and (11), $r_{Q1}^S$ and $r_{Q2}^S$ denote the relative position vectors of the spacecraft as seen from point $Q_1$ and $Q_2$, respectively, whereas $R_\odot$ and $R_{P_i}$ denote the radii of the Sun and the occulting body reported in Table 2. As long as the condition set by Eq. (9) holds, total eclipse occurs whenever Eqs.
and (11) are both satisfied. In contrast, as long as only Eq. (10) is met, the spacecraft is in a situation of partial eclipse. In what follows, we focus on periodic orbits that are close to the surface of the secondary body, as this is the desired configuration for the EQUULEUS (Earth-Moon) and MMX (Mars-Phobos) spacecraft. Hence, the following eclipse analysis neglects the difference between partial and total eclipses.

\[
\text{Table 2: Numerical values used in eclipse analysis.}
\]

<table>
<thead>
<tr>
<th>System</th>
<th>( R_\odot ) (km)</th>
<th>( R_{P_1} ) (km)</th>
<th>( R_{P_2} ) (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth-Moon</td>
<td>696000</td>
<td>6378</td>
<td>1737</td>
</tr>
<tr>
<td>Mars-Phobos</td>
<td></td>
<td>3396</td>
<td>13</td>
</tr>
</tbody>
</table>

2.4. Synodic Resonant Periodic Orbits

The synodic period of a three-body system is defined as the time it takes for the primaries of CR3BP and the Sun to find themselves in the same relative configuration as in Fig. 3 [23]. From Eq. (5), it follows that the synodic period of the Sun-\( P_1- P_2 \) system, \( T_{Sy} \), is

\[
T_{Sy} = \frac{2\pi}{\Omega}
\]

\( T_{Sy} \) is constant as long as the assumptions of Section 2.1 holds. In actuality, the value of \( T_{Sy} \) changes with time in the full-ephemeris model of the same system.
As there is no analytical expression to calculate the synodic period in the full-ephemeris model, $T_{Sy}$ will be conventionally approximated as the average of the synodic periods recorded throughout our numerical simulations. Table 3 reports $T_{sy}$ value for our following analysis.

<table>
<thead>
<tr>
<th>System</th>
<th>Synodic period: $T_{sy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CR3BP full-ephemeris model</td>
<td></td>
</tr>
<tr>
<td>Earth-Moon (day)</td>
<td>29.4873</td>
</tr>
<tr>
<td>Mars-Phobos (hr)</td>
<td>7.6602</td>
</tr>
</tbody>
</table>

SR-PO are periodic orbits whose period, $T_P$, is resonant with $T_{Sy}$. In this paper, two integers $n$ and $m$ are introduced as the numerator and denominator of the synodic ratio $n:m$. According to this definition, a SR-PO completes $m$ revolutions during $n$ synodic periods, such that

$$T_P : T_{Sy} = n : m \quad (13)$$

Since $T_{Sy}$ is considered constant, the period of SR-PO is automatically determined as soon as a synodic ratio is specified.

The advantage of SR-PO is that these periodic orbits are not only periodic in the CR3BP frame of the two primaries, but also in the Body-Sun rotating frames defined in Sec. 2.2. As a result, the eclipse history of any SR-PO spacecraft repeats over time and every $nT_{Sy}$ synodic periods. This implies that the eclipse history of a spacecraft in a SR-PO can be analyzed over $n$ synodic periods rather than for the nominal duration of the mission. During this time frame, the satellite completes $m$ revolutions along the considered periodic orbits in the synodic frame of the two primaries.

Following Eq. (13), SR-PO emerge discontinuously from family branches of periodic orbits, because they depend on the choice of two integers. Figures 4 and 5 show the distribution of candidate SR-PO in the Earth-Moon system, along with an example of a synodic resonant and non-synodic resonant halo orbits in the Moon-Sun rotating reference frames, respectively.
Figure 4: Candidate SR-PO in the Earth-Moon system L2. Only twelve SR-PO (dashed lines) along four periodic orbit families (solid lines) are plotted for the visualization purposes.

(a) $1:3$ SR-HO, $T_P = 9.83$ day.
(b) non SR-HO, $T_P = 9$ day.

Figure 5: Comparison of (a) SR-HO and (b) non SR-HO. The red squares denote points of $z = 0$ crossing, useful for eclipse analyses. Propagation time is 100 days.
3. Eclipse Minimization

In this section, we introduce critical eclipse points as the points along a candidate SR-PO where the spacecraft experiences the longest eclipse duration. This definition follows from the hypothesis that eclipse events—usually lasting for several minutes—may be reduced to single points for design purposes (e.g., z-crossing for spatial orbits, and slowest points for co-planer orbits). Examples of critical eclipse points for a 1:3 SR-HO are shown as red squares in Fig. 5. The key question that we are trying to address here is how to place these eclipse points outside of shadow cones of the two primaries so as to minimize or avoid eclipses. To this end, we derive a geometrical relationship between the longitude of the critical eclipse points in the \( P_2 \)-Sun rotating reference frames and orbit design parameters such as the elongation angle, \( \Theta_0 \), the synodic ratio of a candidate SR-PO, \( n \) and \( m \), and the phase angle, \( \tau_0 \) measured from the point of closest approach. The following derivation focuses on eclipse points with respect to the secondary body, \( P_2 \). However, similar developments may be derived for the other primary of the CR3BP system.

3.1. Longitude of eclipse points

Let us denote \( \psi_j \) as the longitude of the \( j \)-th eclipse point with respect to the \( P_2 \)-Sun rotating coordinate frame. Recalling that a spacecraft in a SR-PO completes \( m \) revolutions in \( n \) synodic months, one finds that the set of all critical eclipse longitudes is given by

\[
\Psi = \bigcup_{j=1,\ldots,N} \left\{ \psi_j + \frac{2\pi n}{m} k \right\}, \quad \forall k = 0, 1, \ldots, (m - 1), \tag{14}
\]

or

\[
\Psi = \bigcup_{j=1,\ldots,N} \left\{ \psi_j + \frac{2\pi m}{m} k \right\}, \quad \forall k = 0, 1, \ldots, (m - 1), \tag{15}
\]

after modulation per \( 2\pi \). The integer \( N \) stands for the number of critical eclipse points and depends on the family of periodic orbit, as well as on the reference frame of choice.
Eclipses can be avoided or minimized with respect to body $P_2$ as long as

$$d \sin \psi_* > R_{P_2},$$

where $\psi_*$ is the closest longitude to $\pi$ among all of the values in $\Psi$; $R_{P_2}$ is the radius of the occulting body; $d = \|r_{S}^{P_2}(\psi_*)\|$ is the distance between the occulting body and the spacecraft when passing through the longitude $\psi_*$. Eq. (16) is a conservative relationship that does not take into account the conical shape of the umbra zone in the attempt of predicting trajectory variations due to the full-ephemeris optimization procedure of Sec. 5.

Observe that the distance $d$ depends on the particular SR-PO being considered (therefore, on our particular selection of the synodic ratio $n : m$), whereas $\psi_*$ depends on the synodic ratio, the initial elongation angle, and the initial insertion point along the candidate SR-PO. In fact, since $r_\odot >> r_{P_2} >> r_{P_1}$, Eq. (8) can be well approximated via

$$\hat{X}_{P_2}^{\odot} \simeq \hat{X}_{\odot}, \quad \hat{Y}_{P_2}^{\odot} \simeq \hat{Y}_{\odot} = \hat{Z}_{P_2}^{\odot} \times \hat{X}_{\odot}, \quad \hat{Z}_{P_2}^{\odot} = \hat{z},$$

so that

$$\psi_j = \arctan \left( \frac{r_{P_2}^{P_2}(\psi_j) \cdot \hat{Y}_{P_2}^{\odot}}{r_{P_2}^{P_2}(\psi_j) \cdot \hat{X}_{P_2}^{\odot}} \right),$$

$$\simeq \theta_j - \Theta_0 + \Omega t_j.$$  

(18)

In Eq. (18), $\theta_j$ is the longitude of the spacecraft at the eclipse point as seen from the barycenter of the body $P_2$:

$$\theta_j = \arctan \left( \frac{y_j}{x_j - (1 - \mu)} \right),$$

(19)

Meanwhile, $t_j$ is the time elapsed between orbit insertion and the time when the spacecraft passes through the critical eclipse location. Without loss of generality, let us rewrite $t_j$ as a function of phase angle $\tau$ introduced in [24]:

$$\tau = 2\pi \frac{t}{T_p},$$

(20)

via

$$t_j = T_p \left( \frac{\tau_j - \tau_0 + 2l\pi}{2\pi} \right),$$

(21)
where $\tau_0$ is the phase angle of the insertion point, and

$$l = \begin{cases} 
0 & \tau_j \geq \tau_0, \\
1 & \tau_j < \tau_0.
\end{cases} \tag{22}$$

From Eq. (12) and (13), it follows

$$\psi_j = \theta_j - \Theta_0 + \frac{n}{m} (\tau_j - \tau_0 + 2l\pi), \tag{23}$$

thereby manifesting the dependency of the desired longitudes on the aforementioned design parameters. Figure 6 illustrates the relevant quantities of interest for eclipse minimization with respect to $P_2$ assuming $\tau_0 = 0$ rad (i.e., orbit insertion at perilune). It is worth noting that the values of $\theta_j$ and $\tau_j$ are immediately determined from their corresponding family branches once a synodic ratio has been specified (see, for example, the values of $\theta_1$ and $\tau_1$ as a function of the orbital period for halo orbits of the Earth-Moon system shown in Fig. 7).

![Figure 6: Eclipse analysis with respect to $P_2$. It is assumed that the spacecraft inject into its SR-PO at the point of closest approach (i.e., $\tau_0 = 0$ deg).](image-url)
Mission designers can pick synodic ratios (corresponding to different orbital periods), insertion epochs (corresponding to different elongation angles), and insertion points (corresponding to different phase angle) in order to minimize the eclipse time of a spacecraft over $n T_{sy}$:

$$\min_{n,m,\Theta_0,\tau_0} \varepsilon_{P_2}(\psi_j(n:m,\Theta_0,\tau_0))$$  \hspace{1cm} (24)$$

where $\varepsilon_{P_2}$ stands for the longest eclipse duration due to $P_2$ measured over $n$ synodic periods. An important distinction is arises, depending on whether the spacecraft trajectory lies completely in the orbital frame of the primaries or not. Secs. 3.2 and 3.3 illustrate how Eq. (23) may be applied to either spatial (e.g., halo and Lyapunov vertical family members) or co-planar orbits (e.g., DROs and Lyapunov planars). In Sec. 4, we will introduce a design tool–hereby referred to as Eclipse Map in order to identify optimal combinations of $\Theta_0$ and $\tau_0$ for eclipse minimization. The interested reader may find additional developments based on Eq. (23) in the Appendix A and B.

### 3.2. Spatial orbits

In the case of three-dimensional synodic resonant periodic orbits, eclipses may be found only when the spacecraft is intersecting the orbital plane of the two primaries whilst in the shadow of either of the two bodies. Intuitively,

![Figure 7: $\theta_1$ and $\tau_1$ values as a function of the Earth-Moon L2 halo orbit period.](image)
critical eclipse points for spatial SR-PO may be identified as the points along
the candidate orbits where \( z = 0 \) plane. It follows that eclipses may be avoided
for the entire duration of the mission as long as the points crossing the \( z = 0 \) plane are never passing through the umbra zones of \( P_1 \) and \( P_2 \). Figure 8 shows an example of an eclipse point for a 1:3 SR-HO with respect to different reference frames. From \( n = 1 \) and \( m = 3 \), we obtain \( \theta_1 = 82.570(\text{deg}) \) and \( \tau_1 = 20.506(\text{deg}) \). Hence, by substituting \( (\Theta_0, \tau_0) = (0, 0) \) into Eq. (23), the longitude of the first eclipse point with respect to the Moon is calculated: \( \psi_1 = 89.405(\text{deg}) \) as in Fig. 8(b). We only show the first eclipse point respect to the Moon for illustration purposes, but other points may be calculated based on Eq. (15).
3.3. Co-planar orbits

Differently from spatial trajectories, co-planar orbits will always experience eclipses throughout any synodic revolution. However, the duration of these eclipse events may be minimized when the points of slowest velocity (with respect to the corresponding $P_i$-Sun rotating reference frame) are not passing through the shadows of either of the two primaries. Following Coriolis’ theorem, the speed of a co-planar satellite ($\dot{z} = 0$) observed from the $P_i$-Sun rotating...
reference frames may be defined as

$$v_{\odot}^{P_i} = \sqrt{(\dot{x} - \Omega y)^2 + (\dot{y} + \Omega (x - x_{P_i}))^2}. \quad (25)$$

The critical eclipse points for candidate planar SR-PO can be then identified as the set of local minima over $\tau \in [0, 2\pi)$:

$$\min_{\tau \in [0, 2\pi]} v_{\odot}^{P_i}(\tau). \quad (26)$$

Figure 9 illustrates the first eclipse point of a 1:2 SR-DRO with respect to different reference frames. From $n = 1$ and $m = 2$, we obtain $\theta_1 = 97.124(\text{deg})$ and $\tau_1 = 81.475(\text{deg})$. Substituting $(\Theta_0, \tau_0) = (0, 0)$ into Eq. (23), the longitude of the first eclipse point with respect to the Moon becomes $\psi_1 = 137.861(\text{deg})$. This value has been confirmed by the numerical simulation of Fig. 9(b).

![Figure 9: Eclipse point transition from (a) CR3BP to (b) Moon-Sun rotating frame. The red square indicates the first eclipse point. Dash lines show a single revolution.](image)

4. Eclipse Maps

In the Sec. 3, we focus on eclipse points with respect to the secondary body. This section considers both primaries and let us denote the variables described in the above with superscript $P_i$, where $i = \{1, 2\}$, for example $\psi_{P_1}^{P_2}$. According
to Eqs. (15) and (23), the set $\Psi^P_i$ is completely determined once the first eclipse point and its angular separation with respect to the remaining $N^P_i - 1$ points along a candidate SR-PO are specified. It is implied that eclipse minimization can be achieved by varying $\psi^P_i$ over $[0, 2\pi]$ as illustrated in Fig. 10 for different SR-HO of the Earth-Moon system. Practically speaking, however, we can control the initial longitude of the first eclipse point by selecting appropriate pairs of $\Theta_0$ and $\tau_0$ values as disclosed in Eq. (23). Figures 11 and 12 demonstrate how the first eclipse longitude is affected by changing the initial value of the elongation angle (corresponding to different insertion epochs), as well as of the insertion point along the SR-PO (corresponding to different values of $\tau_0$).

Figure 10: Candidate SR-HO in the Earth-Moon system. Because of the synodic resonance, the total eclipse time as a function of $\psi^P_1$ is periodic every $2\pi/m$. We set 120 min as a maximum value of eclipse duration for colormap in this figure. Mission designer, however, can set their desired value depending on scientific and engineering constraints.
Figure 11: Effect of changing elongation angle $\Theta_0$. The first revolution is highlighted in blue, whereas the red square denotes the first eclipse point. Magenta star refers to initial point. Black trajectory corresponds to ($\Theta_0$, $\tau_0$)=(0, 0) case. Green trajectory is a single revolution in Moon-Sun rotating frame.
As expected, $\psi_{P1}^P$ changes linearly with $-\Theta_0$, decreases linearly with $\frac{n}{m}(\tau_{P1}^P - \tau_0)$, and jumps by $2\pi n/m$ as soon as $l$ switches from 0 to 1 in Eq. (23).

The situation is better illustrated on a two-dimensional plot, hereinafter referred to as Eclipse Map, for all of the possible values of $\Theta_0$ and $\tau_0 \in [0, 2\pi] \times [0, 2\pi]$. Figures 13 and 14 display the longest eclipse duration of a satellite in a 1:3 SR-HO of the Earth-Moon system. In particular, Fig. 13 shows the longest eclipse duration due to the Moon and Earth, respectively, whereas Fig. 14 displays the values of $\varepsilon_{P12}$. 

Figure 12: Effect of changing phase angle $\tau_0$. The first revolution is highlighted in blue, whereas the red square denotes the first eclipse point. Magenta star refers to initial point. Black trajectory corresponds to $(\Theta_0, \tau_0)=(0, 0)$ case. Green trajectory is a single revolution in Moon-Sun rotating frame.
As it can be seen, for any arbitrary value of $\Theta_0$ (e.g., $\Theta_0 = 0$ deg), there exists a set of insertion points (e.g., $\tau_0 = 0$ deg) for which eclipses are always avoided. Similarly, for any insertion point, it is always possible to find an insertion epoch for which eclipses are minimized. In the Sec. 5, we will investigate how these features can be retained in the full-ephemeris model of the Earth-Moon and Mars-Phobos systems to design minimum-eclipse orbits for scientific missions like EQUULEUS and MMX.
5. Applications

In order to validate our methodology and design tools, this section explores the generation of minimum-eclipse trajectories in the full-ephemeris model of the Earth-Moon and Mars-Phobos systems, respectively. Our choice is motivated by the science orbit design of two JAXA missions that will venture into deep space over the next few years. The first of such missions is the 6U Cube-Sat EQUULEUS, heading to a near-rectilinear halo orbits of the Earth-Moon system, whereas the second mission, MMX, is an international effort lead by JAXA to retrieve samples from the surface of Phobos. Both of these missions will need to conduct key measurement campaigns while coping with tight engineering and scientific constraints. In particular, eclipse minimization is among the top priorities due to its ramifications for the survival of each spacecraft.

Based on this requirement, we hereby consider the design of new scientific orbits enabled by synodic resonant trajectories. These new candidate solutions are presented in Sec. 5.2 and 5.3 after a brief introduction on the strategy and software packages utilized for the full-ephemeris analyses.

5.1. Transition to Full-ephemeris Models

To obtain high fidelity SR-PO, a trajectory optimization software known as jTOP is hereby considered [25]. Figure 15 depicts the continuation strategy of jTOP for transitioning CR3BP periodic orbits into continuous full-ephemeris trajectories. More specifically, key trajectory nodes (e.g., perilune and apolune) are passed to the numerical optimization procedure, which then optimizes the location of these points until their forward and backward propagation match within some low thresholds. For the following analysis, nodes are initially picked along the positive x-axis of the CR3BP frames, and integrated forward and backward in time to overcome numerical instabilities. Figure 16 presents the output of jTOP for a candidate 1:4 SR-HO. It is worth noting that the full-ephemeris trajectories does not need any maneuver for the entire optimization interval, corresponding to six months and one month for the Earth-Moon and Mars-Phobos systems, respectively.
5.2. **EQUULEUS Science Orbit Design**

EQUULEUS is a 6U CubeSat developed by the University of Tokyo in partnership with ISAS/JAXA. The spacecraft will be launched on-board the maiden flight of the Space Launch System (SLS) as one of the thirteen secondary payloads of Artemis-1. Soon after launch, EQUULEUS will separate from SLS and utilize lunar swing-bys to approach the Earth-Moon Lagrangian point L"{2} (EML2). Upon arrival, the CubeSat will insert into a EML2 quasi-halo orbit and start the scientific phase of its mission, observing lunar impact flashes with its instrument suite. During these operations, the spacecraft will also demonstrate
key guidance and station-keeping technologies for future small class satellite missions to the Moon and beyond. However, eclipses shall never last for more than 60 minutes in order to not endanger the survival of the spacecraft.

Based on this requirement, eclipse avoidance is a desired feature for the candidate scientific orbits of EQUULEUS. This was not the case for the preliminary scientific orbits proposed in Ref. [26], which is why synodic resonant trajectories have been lately investigated [20, 27]. As usual, candidate trajectories are the result of a trade-off analysis between the stability properties, transfer opportunities and scientific coverage of different SR-HO. Within this framework, the trajectory design team of EQUULEUS has downselected the number of candidate SR-HO to the 1:4 and 2:5 northern and southern halos family members.

The first step in transitioning these candidate solutions into the real-ephemeris model of the Earth-Moon system is to calculate Earth and Moon eclipse maps as described in Sec. 4. Figure 17 (a) shows the Earth-Moon total eclipse map for the southern 1:4 SR-HO, although similar plots can be generated for all of the remaining cases. The insertion epoch and location are defined based on the eclipse map and used to initialize the jTOP continuation procedure. The candidate eclipse-free halo orbit generated under the CR3BP assumptions is shown in Fig. 17 (b). The point in magenta of Fig. 17 (b) indicates that the spacecraft is inserted into apolune at 00:00:00 June 13, 2020 (UTC). Using this initial conditions and epoch, jTOP outputs a continuous trajectory in the full-ephemeris model of the Earth-Moon system, including the gravity of the Sun and a $2 \times 2$ Earth and $8 \times 8$ Moon spherical harmonics model of the primaries. The full-ephemeris solution is shown in Fig. 18, proving that eclipses are avoided throughout the nominal duration of the mission, which is six month.
5.3. MMX Science Orbit Design

MMX is a sample return mission from the Martian Moon Phobos developed by JAXA and international collaborators to be launched in 2024. The main objective of this mission is to collect data to understand the Martian moons’ origin and the solar system evolution. Before approaching Phobos, MMX will
cruise on a interplanetary trajectory and conduct several orbit insertion man-
uevers upon arrival at Mars. Soon after insertion, MMX will be placed on
stable QSO around the Martian Moon Phobos and start its characterization
and observation campaign in preparation for the sampling operations.

Currently five candidate QSOs labelled as ‘QSO-L’, ‘QSO-Lb’, ‘QSO-Lc’,
‘QSO-M’ and ‘QSO-H’ are under consideration based on the instrument suite
of MMX [28]. Figure 19 illustrates the QSO family around Phobos, including
the targeted orbits for the baseline operations of the spacecraft. Depending
on the insertion epoch, MMX may find itself in a situation where both Mars
and Phobos eclipses can be encountered. Accordingly, it is worth investigating
eclipse-minimum alternatives that would enable the proximity operations of the
spacecraft during this months of the year.

![Figure 19: Candidate SR-QSO in the Mars-Phobos system.](image)

From a SR-PO design point of view, it is found that the 3:4 SR-QSO is the
closest orbit with respect to the QSO-La candidate. Other candidates are either
not close SR-PO or violating minimum eclipse condition stated in Appendix B.
Hence, we generate an Eclipse map for the 3:4 trajectory and present the results
of our calculations in Fig. 20.
As for the EQUULEUS example, selecting a minimum eclipse point in the map corresponds to identify a minimum-eclipse trajectory in the full-ephemeris model of the Mars-Phobos system, 7:50:00 September 2, 2026 (UTC). The initial CR3BP guess and jTOP outputs are shown in Fig. 21, revealing how eclipse times can be minimized over one month regardless of the proximity of the considered trajectory.

Note that the minimum eclipse points identified in the total eclipse map correspond to optimal insertion epochs and locations for which Mars and Phobos eclipses overlap. This interesting feature is illustrated in Fig. 22 where two eclipse histories for a spacecraft in a 3:4 SR-QSO are presented. In the left panel, the satellite is inserted with a \((\Theta_0, \tau_0)\) pair for which the longest eclipse duration is more than 3 hours. In contrast, the right panel shows the eclipse history whenever MMX is inserted into a 3:4 SR-QSO on the red dot of Fig. 20. Comparing with the worst eclipse duration, Phobos eclipses are hereby minimized and perfectly overlapping with Mars occultations.

Figure 20: Mars + Phobos Eclipse map for 3:4 SR-QSO.
Figure 21: 3:4 SR-QSO in Phobos-Sun rotating frame. The shape of Phobos is assumed spherical for eclipse analyses, even though the shape of the Martian moon is highly irregular as shown in this Figure.

6. Conclusions

This paper investigates the exploitation of synodic resonant periodic orbits for minimizing eclipse times under three-body and real-ephemeris dynamics. The main advantage of synodic resonant periodic orbits is that satellite motion becomes periodic in the body-Sun rotating reference frames of the two primaries, drastically simplifying eclipse analyses. Indeed, the entire eclipse history of a spacecraft in a synodic resonant orbit can be reduced to the study of sporadic occultations over a few synodic months. During this time frame, eclipses may be minimized via the appropriate selection of synodic ratios, insertion epochs, and location points, as emphasized by our mathematical developments. Based on these derivations, this paper introduced two-dimensional eclipse maps that allow mission designers to identify optimal initial conditions for minimum-eclipse trajectories. The proposed methodology was tested and verified in the full-ephemeris model of the Earth-Moon and Mars-Phobos systems, producing new candidate science orbits for the upcoming JAXA missions EQUULEUS and MMX. Future work will extend the proposed methodologies beyond the periodic
orbit case, in the attempt of defining optimal eclipse conditions for more general and complicated mission scenarios.

Appendix A. Minimum Eclipse Condition

Regardless of the $\Theta_0$ and $\tau_0$ values in Eq. (23), there exist synodic ratios for which minimum eclipse orbits cannot be found. To elaborate on this statement, let us denote $\Delta \psi^P_{\text{max}}$ as the maximum separation in longitude among all of the elements $\psi_j^P \in \Psi^P_i$. Let us also assume $N^P_i = 2$ as in the case of the spatial periodic orbits considered in this research. By inspection of Eq. (15), it follows that $\Delta \psi^P_{\text{max}}$ must take values in $[\frac{\pi}{m}, \frac{2\pi}{m}]$ depending on the relative longitude between the first and second eclipse points. Let

$$\Delta \psi^P_{12} = \psi^P_{2i} - \psi^P_{1i} = \frac{n}{m} (\tau^P_{2i} - \tau^P_{1i}) + (\theta^P_{2i} - \theta^P_{1i})$$  \hspace{1cm} (A.1)

be this relative longitude, so that

$$\Delta \psi^P_{\text{max}} = \begin{cases} 
\frac{\pi}{m}, & \text{if } \Delta \psi^P_{12} = (2p + 1) \frac{\pi}{m}, \forall p \in N^+, \\
\frac{2\pi}{m}, & \text{if } \Delta \psi^P_{12} = (2p) \frac{\pi}{m}, \forall p \in N^+, \\
\max \left( \left| \Delta \psi^P_{12} - \frac{2\pi q}{m} \right|, \left| \Delta \psi^P_{12} - \frac{2\pi (q + 1)}{m} \right| \right), & q = \left\lfloor \frac{\Delta \psi^P_{12}}{2\pi} m \right\rfloor, \text{otherwise} 
\end{cases}$$  \hspace{1cm} (A.2)
According to Eq. (23), it is always possible to pick a pair \((\Theta_0, \tau_0)\) such that the negative \(x\)-axis of the \(P_i\)-Sun rotating reference frame becomes the bisector of \(\Delta \psi_{P_i}^{\text{max}}\). However, if

\[
d_1 \sin \left( \frac{\Delta \psi_{P_i}^{\text{max}}}{2} \right) + d_2 \sin \left( \frac{\Delta \psi_{P_i}^{\text{max}}}{2} \right) \leq 2 R_{P_i} \tag{A.3}
\]

there must always be a critical eclipse point for which the eclipse minimization condition given by Eq. (16) is violated. Figure A.23 reports the values of \(d \sin \left( \Delta \psi_{P_i}^{\text{max}}/2 \right)\) across the several SR-HO of Fig. 4, noting that \(d_1 = d_2 = d\).

It is found that Moon eclipses cannot be minimized for near-rectilinear halo orbit such as the 2:9 SR-HO. Such an orbit is currently being considered as the baseline for the upcoming Lunar Orbital Platform Gateway [29].

Figure A.23: Eclipse minimization criterion of SR-HO regarding to the Moon.

### Appendix B. Overlapping Phenomenon

Another interesting phenomenon is observed whenever \(N_{P_i} = 2\) and \(\Delta \psi_{12}^{P_i} \approx (2p)\pi/m, \forall p \in N^+\). In this case, eclipse points are found to overlap in the \(P_i\)-Sun rotating reference frame, thereby maximizing our chances to obtain eclipse-minimal trajectories. Figure B.24 shows two examples of overlapping SR-HO as seen with respect to the Moon-Sun rotating reference frame. For the 1:4 and
3:10 cases, $\Delta \psi_{12}^P$ is approximately equal to 90 and 72 deg, respectively. As a result, trajectory legs display wider angular separations with respect to nearby non-overlapping cases such as the 2:7 and 3:11 SR-HO disclosed in Fig. B.25.

![Figure B.24: Overlapping SR-HO as seen from top of the Moon-Sun rotating reference frame. Red and blue squares indicate adjacent eclipse points, respectively.](image1)

![Figure B.25: Non-overlapping SR-HO as seen from top of the Moon-Sun rotating reference frame. Red and blue squares indicate adjacent eclipse points, respectively.](image2)
In the case of retrograde relative trajectories around the Moon or Phobos\cite{30,31}, a more stringent condition can be found. For this type of orbit, $\theta_2^{P_2} = \tau_2^{P_2} \simeq 3\pi/2$, and $\theta_1^{P_2} = \tau_1^{P_2} \simeq \pi/2$, giving $\Delta\psi_{12}^{P_2} = \left(\frac{n+m}{m}\right)\pi$ or $\left(\frac{m-n}{m}\right)\pi$ if the value is shifted over the $[0, 2\pi]$ domain. In order for $\Delta\psi_{12}^{P_2}$ to be equal to $2\pi/m$, $\Delta\psi_{12}$ must be an even multiple of $\pi/m$. In other words, both $m > n$ and $n$ must be odd numbers. Figure\ref{fig:B.26} displays a comparison between an overlapping 3:5 and non-overlapping 4:7 SR-QSO around Phobos, thereby illustrating the main advantage of the former type of orbit. Such a feature turned out to be quite useful when designing low-altitude MMX trajectories around Phobos as in Sec. 5.

![3:5 SR-QSO](image1)

![4:7 SR-QSO](image2)

(a) 3:5 SR-QSO.

(b) 4:7 SR-QSO.

Figure B.26: An overlapping 3:5 and non-overlapping 4:7 SR-QSO orbits around Phobos. The first revolution is highlighted in red. Notice that eclipses cannot be minimized in the latter case.

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